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Deviation of $1/f$ Voltage Fluctuations from Scale-Similar Gaussian Behavior

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Recent measurements on thin metal films suggest a pulse model of resistance fluctuations in which scale similarity and power law spectra are only approximate. We show that such a pulse model is consistent with stationary Gaussian resistance fluctuations. This is to be contrasted with the phenomenological behavior of fluctuations near phase transitions and in turbulent fluids where the fluctuations are non-Gaussian, but exhibit scale similarity of deep physical origin. We then critically examine other tests of the Gaussian behavior of the fluctuating voltage $V(t)$ across a resistor. These include the relaxation of the conditional mean $\langle V(t) \mid V(0) = V_0 \rangle$, and the spectrum of $V^2(t)$. We consider also the question of time reversal invariance. We further ask under what conditions $1/f$ noise can be measured through fluctuations of the Johnson noise power with no applied voltage. We emphasize that this possibility, suggested and observed by Voss and Clarke, requires that $V(t)$ contain a non-Gaussian component.

KEY WORDS: $1/f$ noise; scale similarity; resistance fluctuations; non-Gaussian effects; equilibrium voltage fluctuations; stochastic models.

1. INTRODUCTION

A variety of physical systems exhibit fluctuations whose power spectrum goes approximately as $1/f$ down to the lowest frequencies, f, accessible to experiment. Some mathematical description is possible, but there is little physical understanding. A good introduction and brief review has recently been given by Press.[1] The simplest and best studied example of this "$1/f$

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noise" is the low-frequency voltage fluctuations in conducting materials through which a constant current is flowing. The ubiquity of the phenomenon, and its similarity in a wide range of materials, has suggested that there might be a universal underlying mechanism.[2] The apparent universality of $1/f$ noise might, however, be a rather misleading feature. Not only can many physical mechanisms lead to a given power spectrum, but there is more to a random process than its power spectrum. To discriminate among possible physical mechanisms it can be useful to consider more refined statistical aspects of the resulting noise process. In this paper we concentrate on this aspect of the problem, which has received little attention in the $1/f$ noise literature.

We first contrast statistical properties of $1/f$ noise with two other types of physical problems where power law spectra are observed: critical phenomena and turbulence. The differences which show up at the statistical level (and the possibly only approximate scale similarity of $1/f$ noise) should make one beware of the analogies which a superficial comparison of the power spectra might suggest.

In Section 3 we comment on the presumed Gaussian behavior of resistance fluctuations, and various tests of this behavior. We assume that an experiment is probing resistance fluctuations when the observed spectrum of voltage fluctuations is proportional to the square of the constant applied dc voltage. The question of "time reversal invariance" or "detailed balance" can also be formulated in statistical terms. We discuss the experimental evidence concerning this question.

Other measurements observe $1/f$ noise through slow fluctuations in the Johnson noise power with no applied voltage.[3] In Section 4 we discuss these "equilibrium" measurements, which require that the equilibrium voltage fluctuations be non-Gaussian. We exhibit the correlation function which is measured, state the conditions necessary for the effect to be observable, and discuss how non-Gaussian equilibrium voltage fluctuations may be consistent with Gaussian resistance fluctuations.

We think that this collection of remarks should bring to the reader's mind some connections which have not often been appreciated, and we also hope it will clear up a few points which seem to have been misunderstood in the past.

2. $1/f$ NOISE VERSUS OTHER PHYSICAL PROBLEMS WITH SCALE SIMILARITY

Consider a stationary random process $V(t)$ with mean zero and spectrum

$$S_V(\omega) = C\omega^{-\alpha} \tag{1}$$

between a low-frequency cutoff ω_0 and a high frequency cutoff ω_1. The cutoffs are widely separated and assumed due to known physical causes. We concentrate our attention on the frequency region $\omega_0 \ll \omega \ll \omega_1$ where Eq. (1) is assumed to apply. We call a random process with these properties scale similar. (We use this terminology to distinguish from the well-developed mathematical theory of self-similar random processes. If we introduce high- and low-frequency cutoffs, we need no new probability theory.) If $V(t)$ is Gaussian, then its physical content is fully specified by the scaling exponent α. If not we must give further information to characterize it completely. Many problems of basic interest in statistical physics can be described in terms of such scale similar random processes.

As a first example consider density fluctuations near a gas–liquid critical point. We consider the wave number (k) spectra of equal time spatial correlation functions. Most theoretical descriptions are not manifestly probabilistic, but the probabilistic approach has been developed,[4] and it leads to a description of critical fluctuations as a scale similar random process. The high-wave-number cutoff is determined by a molecular length a, and the low-wave-number cutoff by a dynamically determined correlation length ξ which diverges at the critical point. The scale similarity is most directly manifest in the spectrum of density fluctuations which goes as $k^{\eta-2}$ in the range

$$\xi^{-1} \ll k \ll a^{-1}$$

The critical exponent η is in principle directly observable. A local energy variable, quadratic in the density, also shows critical fluctuations. The spectrum of energy fluctuations in the same wave number range goes as k^{-z}, where z can be related to the usual critical exponents. The fluctuating density field defines a non-Gaussian random process. A characteristic signature of this non-Gaussian behavior is that the exponents η and z are not simply related.

As a second example consider fully developed hydrodynamic turbulence. Again we consider spatial fluctuations. Here the velocity field of the fluid is the random process, and the detailed statistical behavior is directly observable. The underlying dynamical theory is the Navier–Stokes equations of hydrodynamics, but a basic understanding of the phenomena starting from these equations is still lacking. There is, however, a reasonably good phenomenological understanding based on probabilistic description,[5] and we are dealing once more with a scale similar random process. In this case the low-wave-number cutoff is determined by the energy containing length scale of the flow, and the high-wave-number cutoff by the Kolmogorov microscale determined by the direct effects of viscous dissipation. The basic dynamical variable is the vorticity. The scaling exponent determining the spectrum of its fluctuations is directly related to the famous Kolmogorov 5/3 law for the energy spectrum. More recent theory and experiment suggest that the process is non-Gaussian. This shows up most directly in the fluctuation spectrum of the local energy dissipation,[5] which is a variable quadratic in the local vorticity. This spectrum defines a universal exponent for which theoretical calculations do not yet exist, but the random process shows remarkable similarities at the phenomenological level to the density fluctuations near a critical point.[6]

By contrast consider low-frequency voltage fluctuations. Observations in agreement with Eq. (1), over a frequency range, with $0.9 \lesssim \alpha \lesssim 1.4$ are sometimes taken as a *definition* of $1/f$ noise. For example, equilibrium temperature fluctuations lead to low-frequency noise which is Gaussian with a spectrum determined by the dynamics of heat diffusion.[7] Although these fluctuations can appear scale similar when measured over a narrow enough frequency range, they do not correspond in any deep physical sense to a scale similar random process, except in the case of certain singular geometries,[8] or in general at high frequencies. It is generally agreed that most observed $1/f$ noise is not due to equilibrium temperature fluctuations. However, even "true" $1/f$ noise may not really be scale similar.

Recently Dutta, Dimon, and Horn[9] have analyzed the temperature dependence of $S_V(\omega, T)$ in metal films, and have suggested that a thermally activated process with a broad distribution of activation energies dominates the low frequency noise. They do not give an explicit physical mechanism, but suggest a scaling law by which the temperature and frequency dependence of $S_V(\omega, T)$ can be related. Their basic result can be rephrased in the form

$$\left[\omega S_V(\omega, T)/kT\right] = D\left[-kT\log(\omega\tau_0)\right] \tag{2}$$

where τ_0 is a microscopic time of the order of 10^{-14} sec, and the function $D(x)$ has a broad maximum at $x_{\max} \simeq 1$ eV. The function $D(x)$ depends on material, and to some extent also on film thickness. The value of $x_{\max}$ suggests some connection to the dynamics of defects in the films. The effective scaling exponent

$$\alpha(\omega, T) = -\frac{\partial\left[\log S_V(\omega, T)\right]}{\partial(\log\omega)} \tag{3}$$

varies fairly strongly with temperature, but only very weakly with frequency. The random process described by Eq. (2) is not scale similar, but imitates a scale similar process when measured over a frequency range which in practice may be quite wide.

It thus appears to us that the existence of "true" scale similar $1/f$ noise has not been established beyond doubt. A lot of the "unexplained" $1/f$ noises may still be only approximately scale similar. In $1/f$ noise we often see power law spectra over several decades in frequency, but have no theoretical reason to expect scale similarity. This should be contrasted with the situation in critical phenomena and turbulence where the observational basis for scale similarity is weaker, but its theoretical basis is quite sound.

Another difference with the above-mentioned problems is that $1/f$ noises are probably in most instances Gaussian processes. This certainly appears to be the case for resistance fluctuations due to temperature fluctuations. What about the defect mechanism of Dutta, Dimon, and Horn? Is it likely to lead to Gaussian resistance fluctuations? In Appendix A, we consider a pulse model for the fluctuating resistance. Such pulse models are not new,[(10)] and we do not obtain any new results for the spectrum $S_R(\omega)$. We can, however, explicitly look at higher-order statistical quantities. If the average number of pulses present at any time is large, then we find by standard central limit theorem arguments that $R(t)$ is consistent with a Gaussian process. We thus suggest that $1/f$ noise in metal films defines a random process which is Gaussian but not scale similar.

The above conclusion may itself not be universal. There are some older data on carbon resistors which show power law behavior over many decades.[(11)] Deviations from Gaussian behavior are also seen, but these are strongly sample dependent, and suggest that other non-Gaussian noise mechanisms are mixed with the "true" $1/f$ noise in some samples.[(11,12)] Carbon resistors are not, however, very "clean" physical systems. As far as we know there are no measurements of the Gaussian property of $1/f$ noise for metal films, metal whiskers, or any other simpler physical systems where one might expect $1/f$ noise to have an "intrinsic" origin.

Finally, we recall that we have assumed a stationary random process. We have done this strictly for simplicity in the absence of any compelling evidence to the contrary. If nonstationary effects appear at times longer than experimentally accessible, then the observed process can still be described as stationary. If the process is stationary, we can assume that a low-frequency cutoff exists, but is not accessible in a reasonable experimental time. We then avoid any problems associated with an infinite variance. If the mechanism of the noise source involves defect dynamics or other slow forms of structural change, there is no special difficulty in obtaining very long characteristic times. Whether $1/f$ noise is stationary or not may, however, be of importance to understand the physical mechanism involved.

3. GENERAL PROPERTIES OF $1/f$ NOISE: GAUSSIAN BEHAVIOR AND TIME REVERSAL SYMMETRY

How can the Gaussian property of $1/f$ noise sources be tested experimentally? The simplest test is to measure the single-time probability distribution $p(V)$. This can rule out "Gaussianity," but there are many examples in statistical physics of random processes which are not Gaussian, but have Gaussian single-time distributions (for example, the velocity of an atom in a classical fluid at thermal equilibrium[13]). At the level of the two-time distribution, the most direct test involves the moments

$$D_n(t) = \langle [V(t) - V(0)]^n \rangle$$

sometimes called the structure functions of the process. If the two time distribution is Gaussian, we have, for example,

$$D_4(t) = 3[D_2(t)]^2$$

which is equivalent to the frequently quoted property of a stationary Gaussian process

$$\langle V^2(0) V^2(t) \rangle = 2\langle V(0) V(t) \rangle^2 + \langle V^2(0) \rangle^2 \tag{4}$$

This has not been directly checked for $1/f$ noise sources, but an indirect check has been obtained by Stoisiek and Wolf,[14] who measured the variance of the random variable

$$\eta_T(t) = \frac{1}{T} \int_0^\infty \exp(-t'/T) V^2(t-t') dt' \tag{5}$$

as a function of the averaging time T. They found consistency with Eq. (4) for a carbon resistor and a bipolar transistor. In our opinion the direct measurement of $\langle V^2(0) V^2(t) \rangle$, or the essentially equivalent measurement of Stoisiek and Wolf, are the most natural tests of the Gaussian property of $V(t)$. Needless to say, one can never check whether a process is strictly Gaussian since this implies a measurement of an infinite number of higher-order correlation functions. However, for practical purposes, if $\langle V^2(0) V^2(t) \rangle$ satisfies Eq. (4) one can say that the process is Gaussian. It is interesting that in the example of turbulent fluid flow discussed earlier, Eqs. (4) and (5) are just the quantities suggested by Kolmogorov[15] as a measure of the non-Gaussian nature of the velocity field. In that case the local vorticity plays the role of $V(t)$, and its square is the local rate of energy dissipation.

Another possibility to check Gaussian behavior is to study the conditional mean $\langle V(t) \mid V(0) = V_0 \rangle$. This function represents the average value of the voltage at time t given that at time $t = 0$ the voltage had the value V_0. For a stationary Gaussian process this satisfies the condition[16]

$$\frac{\langle V(t) \mid V(0) = V_0 \rangle}{V_0} = \frac{\langle V(0) V(t) \rangle}{\langle V^2(0) \rangle} \tag{6}$$

where the quantities on the right-hand side of this equation are the usual correlation functions. In an interesting paper, Voss[17] has studied the validity of Eq. (6) for a variety of $1/f$ noise sources. He found it satisfied in carbon resistors and field effect transistors, but not satisfied in p–n junction devices. Equation (6) can be thought of as a kind of linearity condition, and Voss suggests that it is a test of the linearity of the underlying noise mechanism. This interpretation of Eq. (6) must be qualified. Equation (6) is automatically satisfied for any stationary Gaussian process. The Gaussian behavior typically arises via the central limit theorem by linear superposition of independent events. Nothing is assumed about the microscopic dynamics of these individual events which could be nonlinear. For example, in the pulse model which we discuss in Appendix A, there is no physical mechanism assumed for the individual resistance pulses. If one looks, however, for a phenomenological description of a process where the observable variable obeys a stochastic differential equation with a Gaussian noise source, then the "linearity" test of Voss may indeed prove useful in deciding whether or not this differential equation is linear.

Voss has also suggested that checking whether the condition $\langle V(t) \mid V(0) = V_0 \rangle = \langle V(-t) \mid V(0) = V_0 \rangle$ is satisfied constitutes a test of time reversal invariance for the process. We would like to qualify that statement slightly: The correct statement should be that if a process is stationary and time reversal invariant, then Voss's condition is satisfied. The converse, however, is not true. In Appendix B we show that there exist stationary processes which satisfy Voss's condition, but which are not time reversal invariant.

A simple test of time reversal noninvariance is the third-order structure function

$$D_3(t) = \left\langle \left[V(t) - V(0) \right]^3 \right\rangle$$

This vanishes if there is symmetry between t and $-t$. Again time reversal invariance is a sufficient but not necessary condition for D_3 to vanish. Testing for $D_3 = 0$ is mathematically an even less stringent test than that suggested by Voss, but it may in practice be easier to perform.

Note that D_3 vanishes for any Gaussian process. This illustrates the more general point that any stationary one-variable Gaussian process is necessarily time reversal invariant. Such a process is fully determined by its correlation function $\langle V(0) V(t) \rangle$, which is an even function of t by stationarity. There has been some confusion on this point in the literature. Press[1] has remarked that $1/f$ noise generated by appropriately filtering Gaussian white noise could have a different arrow of time depending on whether causal or acausal filters are used. This is incorrect since any stationary process generated by linear filtering of a stationary Gaussian process is itself a stationary Gaussian process. Thus it is necessarily time reversal invariant. [To be sure on the question of stationarity we restrict ourselves to band-limited $1/f$ noise which can be generated by a nonsingular filter $G(\omega)$.] Our point can also be seen by a slightly different argument. If $G(\omega)$ is a causal filter, then its complex conjugate $G^*(\omega)$ is the corresponding acausal filter. The spectrum of the filtered process is then $G(\omega)S_0(\omega)G^*(\omega)$ for both causal and acausal filtering, where $S_0(\omega)$ is the spectrum of the unfiltered process. Only if the unfiltered process is not Gaussian can the causally and acausally filtered processes be distinguished. They will both have the same spectrum, but higher-order correlation functions, such as $D_3(t)$, can distinguish them.

4. "EQUILIBRIUM" $1/f$ NOISE: ON NON-GAUSSIAN EQUILIBRIUM VOLTAGE FLUCTUATIONS AND GAUSSIAN RESISTANCE FLUCTUATIONS

Voss and Clarke[3] have suggested that $1/f$ voltage fluctuations can be observed in "equilibrium" systems without applying a voltage. The average Johnson noise power in a frequency band Δf below the RC cutoff in a capacitive circuit is given by $4kTR\Delta f$. They suggest that slow resistance fluctuations should modulate this noise power. Similarly the noise in the band Δf above the RC knee should be proportional to $R^{-1}\Delta f$, and should show slow fluctuations. The total noise power integrated over all frequencies should, however, not depend on the resistance and should not exhibit slow fluctuations. They confirmed this prediction experimentally in small semiconductor and metal films. The same effect has been seen in carbon resistors by Beck and Spruit.[18]

These experiments show that $1/f$ noise is not caused by the applied voltage, but they do not show that it is a thermal equilibrium phenomenon. In many cases the noise mechanism may be associated with very slow structural fluctuations in the material. These are likely to involve subtle forms of metastability in the state of the material which should be distinguished from true thermodynamic equilibrium.

In terms of $V(t)$ as a random process the above effect is manifestly non-Gaussian since a variable quadratic in $V(t)$ exhibits slow fluctuations which do not appear in $\langle V(0)V(t)\rangle$. From the remarks in the first paragraph of this section, we can see that the effect is more subtle than we have discussed earlier, since the slow fluctuations will modify only slightly the simplest four-voltage correlation function $\langle V^2(0)V^2(t)\rangle$. To have observable effects, one must look at

$$\langle V(0)V(\delta')V(t)V(t+\delta)\rangle \tag{7}$$

where δ and δ' are not zero but are much smaller than the time scale t of the slow fluctuations (see also Appendix C).

Beck and Spruit[18] have discussed the conditions for observability of the effect discovered by Voss and Clarke.[3] In Appendix C, we present what we believe to be a simpler and clearer derivation of which correlation function is measured as well as the conditions for observability of the effect. In particular we calculate the level of the background due to Gaussian fluctuations. We find that this background is equal to $1/\Delta f$ where Δf is the bandwidth of the Johnson noise sampled by the experiment.

The Voss–Clarke result is intuitively plausible, and has some experimental confirmation, but there is no microscopic theoretical justification. In the case that the slow fluctuations are due to equilibrium temperature fluctuations, we have a basic physical understanding and should be able to achieve a more microscopic description of $V(t)$ as a random process. This is quite difficult, however, from a systematic point of view since both the Johnson noise and temperature fluctuation noise are of the same order in an expansion in Ω^{-1} where Ω is the volume of the system. The fluctuations in the band-limited Johnson noise power are of higher order in Ω^{-1}, and no systematic theory exists which keeps all terms of this higher order. A reasonable physical guess can be made, however, as to the dominant higher-order effects, and this leads to a modified Langevin equation in which the slow resistance fluctuations modulate both the friction constant, and the intensity of the noise source. We have developed this description in some detail, but will defer its presentation for another paper.

We can summarize our preliminary results as follows. We agree with Voss and Clarke with one minor difference. For equilibrium temperature fluctuations, the amplitude of the noise power fluctuations is proportional to $(\beta T_0)^2$ rather than $(1 \pm \beta T_0)^2$, where $\beta = R^{-1} dR/dT$. In practice this is irrelevant since the effect is only observable if β is very large. We have also calculated the corrections from the slow resistance fluctuations to the Johnson noise spectrum $S_V(\omega)$. These are very small as expected. Nevertheless, our model has the virtue of showing how a small nonlinearity in the equilibrium equations of motion for $V(t)$ leads to its non-Gaussian behavior for the four-point equilibrium correlation and at the same time to a Gaussian behavior in leading order in the nonequilibrium situation.

5. CONCLUSION

We have clarified a variety of questions concerning the statistical description of $1/f$ noise. We have suggested that in clean well-characterized physical systems this noise is not likely to have any scale similarity of deep physical origin, but it is likely to be Gaussian. In metallic systems[9] the most plausible physical mechanism involves defect migration or other slow forms of structural fluctuation. There are then no special difficulties with the long time scales involved, but there are also no detailed mechanisms known which can account for the observations. The main thrust of future research in this area must be experimental, particularly in dealing with well-characterized samples whose material properties can be varied. In such an experimental program it would be valuable to test the scale similarity and Gaussian property of the noise process, and to relate them to material properties. This requires precise measurements of the spectrum over a wide frequency range, and more attention to higher-order correlation functions in order to test the Gaussian property.

APPENDIX A: HIGHER-ORDER STATISTICS OF PULSE MODELS

Pulse models have been considered, for example, by Halford.[10] Consider a sequence of N pulses in the interval $-T/2 \leqslant t' \leqslant T/2$, and take $N/T = W = \text{const}$. The onset of each pulse is described by a set of random variables s_i^τ which follow a Poisson process.[4] Then, let the resistance of the sample be

$$R(t') = \sum_{\tau=\tau_{\min}}^{\tau_{\max}} \sum_{i=1}^{N_\tau} f_\tau(t' - s_i^\tau) \tag{A1}$$

where $f_\tau(t)$ is a function which is parametrized by the parameter τ which describes the interval of time during which the function $f_\tau(t)$ differs from zero. Let N be the total number of random variables s_i^τ in the interval $-T/2 \leqslant t' \leqslant T/2$. Then,

$$\sum_{\tau=\tau_{\min}}^{\tau_{\max}} N_\tau = N \tag{A2}$$

We interpret the ratio N_τ/N as the probability of having a pulse whose duration will be τ.

To find higher-order statistics for the random variable defined by Eq. (A1), we compute the characteristic function:

$$G_q(t) = \langle \exp[iq(R(t_1 + t) - R(t_1))] \rangle \tag{A3}$$

Using Eq. (A1) and the fact that the s_i^τ are independent random variables we find, using stationarity, that

$$G_q(t) = \prod_{\tau=\tau_{\min}}^{\tau_{\max}} \prod_{i=1}^{N_\tau} \langle \exp[iq(f_\tau(t - s_i^\tau) - f_\tau(-s_i^\tau))] \rangle \tag{A4}$$

[4]For a Poisson system of random points, the occurrence of each point is described by an independent random variable. If the Poisson process is stationary, each of these independent random variables has a constant probability density [R. L. Stratonovich, *Topics in the Theory of Random Noise*, Vol. 1 (Gordon and Breach, New York, 1963), pp. 146 and 153].

where the angular brackets are averages over the random variables s_i^τ. For a stationary Poisson process, each of the random variables s_i^τ has a probability distribution, $p(s_i^\tau)\,ds_i^\tau = (ds_i^\tau/T)$.

For concreteness, we will take

$$f_\tau(t) = \begin{cases} 0 & \text{if } t < 0 \\ A_\tau & \text{if } 0 < t < \tau \\ 0 & \text{if } t > \tau \end{cases} \tag{A5}$$

The specific form of $f_\tau(t)$ comes in the final result in such an inessential way that many of our conclusions can certainly be generalized for an arbitrary function $f_\tau(t)$. Using Eq. (A5) in (A4) we find, for $t > 0$,

$$G_q(t) = \prod_{\tau=\tau_{\min}}^{\tau_{\max}} \prod_{i=1}^{N_\tau} \left\{ \left[\frac{t}{T}(e^{iqA_\tau} + e^{-iqA_\tau}) + \frac{T-2t}{T} \right] \Theta(\tau - t) + \left[\frac{\tau}{T}(e^{iqA_\tau} + e^{-iqA_\tau}) + \frac{T-2\tau}{T} \right] \Theta(t-\tau) \right\} \tag{A6}$$

If we assume that $(t/T)N_\tau \ll N_\tau$ and take the limit $N_\tau \to \infty$, $T \to \infty$ keeping $N/T \equiv W$ constant, we obtain, for $\tau_{\min} \leqslant t \leqslant \tau_{\max}$,

$$G_q(t) = \exp\left\{ -2W \left[\sum_{\tau=\tau_{\min}}^{t} \tau \frac{N_\tau}{N}(1 - \cos A_\tau q) + \sum_{\tau=t}^{\tau_{\max}} t \frac{N_\tau}{N}(1 - \cos A_\tau q) \right] \right\} \tag{A7}$$

As mentioned before we should interpret $p(\tau) = N_\tau/N$ as the probability of having pulses of length τ.

Following closely the arguments which are used for the standard proof of the central limit theorem, we can deduce from Eq. (A7) that in the limit where $W\tau_{\min}$ is large, the process becomes Gaussian. Indeed, if $W\tau_{\min}$ is large, the coefficient of each term of a power series in q for $G_q(t)$ is dominated by the contribution from the term of order q^2 in the exponential. This means that to order $(W\tau_{\min})^{-1}$ only the first cumulant need be used to compute all correlation functions, hence the process is Gaussian. In the Gaussian limit Eq. (A7) becomes

$$G_q(t) = \exp\left\{ -Wq^2 \left[\int_{\tau_{\min}}^{t} A_\tau^2 p(\tau)\tau\,d\tau + \int_t^{\tau_{\max}} A_\tau^2 p(\tau) t\,d\tau \right] \right\} \tag{A8}$$

The first cumulant is given by the exponent in Eq. (A8) and it will decay logarithmically in time ($1/f$ noise) if $\tau_{\min} \leqslant t \leqslant \tau_{\max}$ and

$$A_\tau^2 p(\tau) \sim \frac{1}{\tau^2} \tag{A9}$$

This condition has been emphasized by Halford.[10] Note that we recover the McWorther model, $p(\tau) \sim 1/\tau$, if we notice that in that model, each of the pulses has a correlation function normalized to the same value at $t = 0$ independent of τ. This can be achieved in our model only if $A_\tau^2 \sim 1/\tau$ as can be seen from considering the coefficient of order q^2 of the following characteristic function:

$$G_q^\tau(t) = \exp\{-2W(1 - \cos A_\tau q)[\tau\theta(t - \tau) + t\theta(\tau - t)]\} \quad \text{(A10)}$$

It is still possible that non-Gaussian features of the process $R(t)$ could appear in multitime correlation functions such as $\langle R(t_1)R(t_2)R(t_3)R(t_4)\rangle \cdot$ $(t_1 \neq t_2 \neq t_3 \neq t_4)$. This question has not yet been examined.

APPENDIX B: ON MODELS WHICH ARE NOT TIME REVERSAL INVARIANT BUT STILL HAVE THE PROPERTY $\langle V(t) \mid V(0) = V_0\rangle = \langle V(-t) \mid V(0) = V_0\rangle$

We will assume that the random variable V can take only a set of N discrete values V_n. We define $P_2(V_m, t; V_n, 0)$ the joint probability for observing the value V_n of the random variable V at time 0 and the value V_m at a time t later. We will assume that the process is stationary; hence P depends only on the time difference between the first and the second measurements. For definiteness, we shall consider a fixed time difference between the measurements and henceforth omit the time label:

$$P_2(V_m, V_n) \equiv P_2(V_m, t; V_n, 0) \quad \text{(B1)}$$

Note that from P_2 we can compute the correlation function

$$\langle V(t)V(0)\rangle = \sum_{n,m}^{N} V_m V_n P_2(V_m, V_n) \quad \text{(B2)}$$

and the conditional means

$$\langle V(t) \mid V(0) = V_n\rangle = \sum_{m}^{N} V_m P_2(V_m, V_n) P_1^{-1}(V_n) \quad \text{(B3)}$$

$$\langle V(t) = V_n \mid V(0)\rangle = \sum_{m}^{N} V_m P_2(V_n, V_m) P_1^{-1}(V_n) \quad \text{(B4)}$$

where $P_1(V_n)$ is the single time probability distribution for V.

We define the property of time reversal symmetry as follows:

$$P_2(V_i, V_j) = P_2(V_j, V_i) \quad \text{(B5)}$$

If a system has "true" time reversal symmetry, the property (B5) must be obeyed for all values of the time difference *and* the higher-order joint probability distributions must also obey obvious generalizations of Eq. (B5). It is possible that certain systems do not obey time reversal symmetry while condition (B5) is still satisfied. For that reason, the condition (B5) is sometimes called "detailed balance."

Note that at this stage, it is already obvious that Voss's condition, which applies to an average, is weaker than the requirement of time reversal invariance, which is a condition on the full distribution. We will see below that if N is large enough indeed more information is contained in Eq. (B5) than in Voss's condition.

Clearly, if time reversal invariance is satisfied [Eq. (B5)], then Eqs. (B3) and (B4) are equal (Voss's condition). On the other hand, in general, it takes N^2 numbers to specify $P_2(V_i, V_j)$. There are a few constraints which P_2 must also satisfy: normalization,

$$\sum_{m,n}^{N} P_2(V_n, V_m) = 1 \tag{B6}$$

and

$$\sum_{m}^{N} P_2(V_m, V_n) = P_1(V_n) \tag{B7}$$

$$\sum_{m}^{N} P_2(V_n, V_m) = P_1(V_n) \tag{B8}$$

by definition of P_1 and P_2. Equations (B6)–(B8), however, represent at most $(2N + 1)$ independent constraints. Even if Eqs. (B3) and (B4) are equal for all values of V_n, that gives us at most N other independent constraints. Thus, for N large enough, we will always be left with enough freedom in the remaining $N^2 - (3N + 1)$ independent values of P_2 to devise a process where time reversal symmetry [Eq. (B5)] is not obeyed even though Voss's condition and the usual constraints on P_2 are satisfied. [Note that when Eq. (B5) is satisfied, we are left with only $N(N + 1)/2$ independent elements even before the constraints (B6) to (B8) are applied.] In practice, N tends to infinity.

We do not want to be purists: one will never be able to check whether a process is exactly time reversal invariant (or for that matter, whether a process is Gaussian) because it would take an infinite number of measurements on higher-order correlation functions. For practical purposes, a few tests to check consistency with time reversal invariance suffice. Equation (B5) is one of the conditions which must be satisfied if there is time reversal invariance. We find it amusing that Voss's condition is mathematically a weaker test of Eq. (B5) than one would have expected, given that it is an intuitively satisfying test of time reversal invariance.

APPENDIX C: CORRELATION FUNCTION FOR "EQUILIBRIUM" $1/f$ NOISE MEASUREMENT

This problem has also been studied by Beck and Spruit.[18] Our results are equivalent to theirs but our perspective is different.

Consider the "filtered" Johnson noise voltage,

$$V_{\omega_0}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} g_\omega V_\omega \tag{C1}$$

where g_ω the "filter" function is peaked around $\omega = \pm\omega_0$. Proceeding like Voss and Clarke we square Eq. (C1) and average over an interval $\Delta t \gtrsim 1/\omega_0$ to obtain the random variable,

$$P(t) = V^2_{\omega_0}(t) = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \left[\frac{\sin(\omega + \omega')\Delta t/2}{(\omega + \omega')\Delta t/2} \right] e^{-i(\omega+\omega')t} g_\omega g_{\omega'} V_\omega V_{\omega'} \tag{C2}$$

Note that,

$$\langle P(t) \rangle = 2 S_V(\omega_0)\, \Delta f \tag{C3}$$

where

$$S_V(\omega_0) = \int_{-\infty}^{\infty} dt\, e^{i\omega_0 t} \langle V(0) V(t) \rangle \tag{C4}$$

To obtain Eq. (C3), we have assumed that $S_V(\omega)$ is slowly varying over the width of the filter Δf:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |g_\omega|^2 \equiv 2\,\Delta f \tag{C5}$$

Note that $g_{-\omega} = g^*_\omega$.

The power spectrum $S_p(\nu)$ of the random variable $P(t)$ is defined by an equation analogous to Eq. (C4). If,

$$\nu\,\Delta t \ll 1, \qquad (1/\Delta t \leqslant \omega_0) \tag{C6}$$

then,

$$S_p(\nu) = \int \frac{d\bar\omega}{2\pi} \frac{d\bar\omega'}{2\pi} \frac{d\omega}{2\pi} \langle V_{\bar\omega} V_{\bar\omega'} V_\omega V_{\nu-\omega} \rangle g_{\bar\omega} g_{\bar\omega'} g_\omega g_{\nu-\omega} \tag{C7}$$

To have a nonzero value of $S_p(\nu)$ we need

$$\nu < \Delta\omega, \qquad \Delta\omega \equiv 2\pi\,\Delta f \tag{C8}$$

otherwise $g_\omega g_{\nu-\omega} \simeq 0$. The conditions (C6) and (C8) are implicitly contained in the work of Beck and Spruit.[18] Condition (C8) is quite natural since each data point for $P(t)$ can be collected only in a time $\delta t \geqslant 1/\Delta\omega$. ($1/\Delta\omega$ is also the correlation time of the sampled Johnson noise.) Since the fastest Fourier component we can measure from these data points has a frequency $\nu \lesssim 1/\delta t$, the condition (C8) follows. Note that condition (C6) ($\nu < \omega_0$) is automatically satisfied if (C8) is satisfied and $\Delta\omega < \omega_0$. If $\Delta\omega > \omega_0$ then $\nu < \omega_0$ is a stronger restriction than Eq. (C8). The inequality $\nu < \omega_0$ comes about because in Eq. (C2) each point is an average over a time Δt. This averaging is quite natural since it eliminates high frequencies from Eq. (C2). This averaging does not, however, appear essential to us since $\Delta t = 0$ is consistent with $\nu\,\Delta t \ll 1$ and Eq. (C7) follows even if $1/\Delta t \gtrsim \omega_0$. There may thus be cases where the Voss and Clarke experiment works even if $\nu > \omega_0$.

If V is a Gaussian random variable, the average in Eq. (C7) may be evaluated. Using Eq. (C3) and the usual Gaussian decomposition we find

$$\frac{S_p(\nu)}{\langle P\rangle^2} \simeq \frac{1}{\Delta f} \tag{C9}$$

In general, this "Gaussian background" is unavoidable. Since each point $P(t)$ can be taken as an "estimate" of the "true" Johnson noise power Eq. (C3), Eq. (C9) can be interpreted as being simply due to the "sampling" error of each estimate. Voss's and Clarke's conjecture is that the non-Gaussian component of the voltage fluctuations can be estimated from

$$\frac{S_p(\nu)}{\langle P\rangle^2} \sim \frac{S_R(\nu)}{R^2} \tag{C10}$$

where the right-hand side of this equation may be obtained from a standard $1/f$ noise experiment. The one experiment of Beck and Spruit on carbon resistors and those of Voss and Clarke on Nb and InSb agree with Eqs. (C9) and (C10). The background in the InSb experiment is larger than the one which can be computed from Eq. (C9) because one of the amplifiers had a smaller bandwidth than that of the original filtered Johnson noise.[19]

Finally, note that Eq. (C7) may be written as a weighted average of correlation functions of the form of Eq. (7) as quoted in the text. Recalling that Eq. (C10) is proportional to the inverse of the size of the system, we also see by comparison of Eqs. (C9) and (C10) that the non-Gaussian effects we are looking at are of higher order in an expansion in powers of the inverse of the volume of the system.

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